

Brownian motion as limit of the interchange process

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Abstract

We prove that the random empirical measure of appropriately rescaled particle trajectories of the interchange process on paths converges weakly to the deterministic measure of stationary Brownian motion on the unit interval. Our approach provides a direct derivation of the hydrodynamic limit of the Simple Symmetric Exclusion process on paths.

1 Introduction

Let L_n denote the finite path of length n , which is the graph on the vertex set $\{1, \dots, n\}$ with edges between every adjacent integers i and $i + 1$. Insert self-loops at the end points 1 and n so that every vertex has degree 2. The interchange process on L_n is defined as follows. Initially, each vertex i in L_n has a particle on it with label i . The particles move at random in continuous time; each of the $n + 1$ edges has an independent Poisson process of rate $1/2$ that indicates the times the particles move. An edge *fires* at time t if there is a point on its Poisson process at time t . Whenever an edge fires the two particles along its endpoints are swapped. We stipulate that the swap occurs instantaneously at the time the edge fires, so if edge $\{i, i + 1\}$ fires at time t then the particle at position i at time t is the particle that was at position $i + 1$ just prior to time t . We may assume that all the times along all the different Poisson processes are distinct so that no two edges fire at the same time. Let $\mathbf{Int}_t^n(i)$ be the position of particle i at time t on L_n . Then $i \rightarrow \mathbf{Int}_t^n(i)$ is a permutation of $\{1, \dots, n\}$ and the interchange process is the permutation valued process $\mathbf{Int}^n = (\mathbf{Int}_t^n; t \geq 0)$.

The convention that particles jump instantaneously as edges fire implies that the trajectory of any particle is a càdlàg path (right continuous with left limits). Also, having the edges fire at rate $1/2$ implies that each particle moves at rate 1. Let

$$T_i^n(t) = \frac{\mathbf{Int}_{n^2 t}^n(i)}{n} \quad \text{for } t \geq 0 \quad (1.1)$$

be the rescaled trajectory of particle i . Consider the random empirical measure

$$\nu^n = \frac{1}{n} \sum_i \delta_{T_i^n}. \quad (1.2)$$

The measure ν^n is a random Borel probability measure on $D(\mathbb{R}_+)$, the space of càdlàg paths from $\mathbb{R}_+ = [0, \infty)$ into $[0, 1]$ with the Skorokhod J_1 -topology. We prove that ν^n converges weakly to a deterministic measure - the law of stationary Brownian motion on $[0, 1]$. The latter is the law of standard Brownian motion started from an uniform random point in $[0, 1]$ and reflected off of the lines $y = 0$ and $y = 1$.

Observe that any weak limit of ν^n is a priori a random measure on $D(\mathbb{R}_+)$. The support of this measure consists of càdlàg process $X = (X(t); t \geq 0)$ that have the additional property that for every t the law of $X(t) \sim \text{Uniform}[0, 1]$. This follows from observing that for every t the law of $(1/n) \sum_i \delta_{T_i^n(t)}$ is always uniform on the set $\{i/n; 1 \leq i \leq n\}$ because $\mathbf{Int}_{n^2 t}^n$ is a permutation. We call such processes X *permuton processes* because they arise within the limit theory of permutation-valued processes [5].

Theorem 1. *Consider the interchange process \mathbf{Int}^n on a path of length n and let ν^n from (1.2) be the empirical measure of the particle trajectories. Then ν^n converges weakly – as a random probability measure on $D(\mathbb{R}_+)$ – to the deterministic measure concentrated on the law of stationary Brownian motion on $[0, 1]$.*

Theorem 1 extends to higher dimensions as well; the interchange process on growing d -dimensional boxes $[1, n]^d$ converges in the sense of Theorem 1 to the law of stationary Brownian motion on $[0, 1]^d$. It is an interesting question as to when the interchange process converges for other sequences of finite graphs, for instance, graph sequences that converge in the Gromov-Hausdorff distance.

Background and Motivation Theorem 1 is a law of large numbers type phenomenon for particle trajectories of the interchange process. Such results have been proved by Rezakhanlou [6] for the Symmetric Simple Exclusion Process (SSEP). Kipnis and Varadhan [3] also have similar earlier work on the SSEP. The proofs for SSEP are based on identifying the limit as the unique solution of an appropriate hydrodynamic equation. Our methodology for the interchange process is based on coupling and derives the limit as a direct consequence of Donsker's Theorem. In Section 4 we explain how Theorem 1 provides the hydrodynamic limit of the SSEP. A motivation for our work is recent work by Kotowski [4], who uses these ideas to prove a large deviation result for the interchange process with particles moving according to location dependent rates. It has applications to limit behaviour of random sorting networks.

2 Tightness and Concentration Criterion

The following lemma provides a criterion for ν^n to be tight. Let $\mathcal{M}(K)$ denote the space of Borel probability measures on a metric space K in the weak topology. For a function $f : [0, T] \rightarrow [0, 1]$ its càdlàg modulus of continuity is

$$m_f(\delta) = \inf_{\Pi} \max_i \sup_{s, t \in [t_{i-1}, t_i)} |f(t) - f(s)|,$$

where the infimum is over all finite partitions $\Pi = \{0 = t_0 < t_1 < \dots < t_k = T\}$ satisfying $\min_i |t_i - t_{i-1}| \geq \delta$. A function f is càdlàg if and only if $m_f(\delta) \rightarrow 0$ as $\delta \rightarrow 0$.

Lemma 2. *Let f_1^n, \dots, f_n^n be a collection of random paths in $D(\mathbb{R}_+)$. Consider the empirical measure of these paths: $\mu^n = \frac{1}{n} \sum_{i=1}^n \delta_{f_i^n}$. Let $m_i^n(T, \delta)$ be the càdlàg modulus of continuity of f_i^n restricted to $[0, T]$. Suppose for every $T < \infty$*

$$\limsup_{\delta \rightarrow 0} \limsup_{n \rightarrow \infty} \frac{1}{n} \sum_i \mathbb{E} [m_i^n(T, \delta)] = 0. \quad (2.1)$$

Then the sequence of random measures $\{\mu^n\}$ is tight. In particular, tightness follows if there is an $m : [0, 1] \rightarrow \mathbb{R}_+$ that satisfies $\lim_{\delta \rightarrow 0} m(\delta) = m(0) = 0$ and

$$\limsup_{\delta \rightarrow 0} \limsup_{n \rightarrow \infty} \max_i \mathbb{P} [m_i^n(T, \delta) > m(\delta)] = 0.$$

Proof. We may replace the limit supremum over n in (2.1) by a supremum over n because for each n and i the expected càdlàg modulus of continuity $\mathbb{E} [m_i^n(T, \delta)] \searrow 0$ as $\delta \searrow 0$.

The sequence $\{\mu^n\}$ is tight if for every $T < \infty$ the empirical measure of the restriction of the paths f_1^n, \dots, f_n^n to $[0, T]$ is tight. Prokhorov's Theorem [1, Theorem 14.3] provides tightness if for any $\epsilon > 0$ there exists a compact $A_\epsilon \subset \mathcal{M}(D([0, T]))$ such that $\sup_n \mathbb{P} [\mu_n \notin A_\epsilon] \leq \epsilon$. A subset $A \subset \mathcal{M}(D([0, T]))$ is compact if for every integer $r \geq 1$ there is a compact subset $K_r \subset D([0, T])$ such that $\sup_{\mu \in A} \mu(K_r^c) \leq 1/r$, which also follows from Prokhorov's Theorem. The compact subsets of $D([0, T])$ in the Skorokhod topology consists of

functions with a common càdlàg modulus of continuity: for any ϵ there is a δ such that $m_f(\delta) \leq \epsilon$ for every function f in the subset. We apply these criteria to construct the required compact subsets A_ϵ .

For fixed $\epsilon > 0$ we may choose $\delta_k \searrow 0$ such that $\sup_n \frac{1}{n} \sum_i \mathbb{E}[m_i^n(T, \delta_k)] \leq \epsilon/k^4$. This is possible due to the assumption in (2.1). Consider the following compact subset of $D([0, T])$:

$$A_{\epsilon, r} = \{f \in D([0, T]) : m_f(\delta_k) \leq r^3/k^2 \text{ for every } k\}.$$

Set

$$A_\epsilon = \{\mu \in \mathcal{M}(D([0, T])) : \mu(A_{\epsilon, r}) \geq 1 - (1/r) \text{ for every } r\}.$$

Then A_ϵ has compact closure in $\mathcal{M}(D([0, T], [0, 1]))$ and it suffices to show that for $\mathbb{P}[\mu^n \notin A_\epsilon] \leq \epsilon$ for every n . To this end, observe that for any subset $A \subset D([0, T])$ we have $\mathbb{E}[\mu^n(A)] = (1/n) \sum_i \mathbb{P}[f_i^n \in A]$. Applying union bounds in r and k respectively, and also Markov's inequality, we get

$$\begin{aligned} \mathbb{P}[\mu^n \notin A_\epsilon] &\leq \sum_r \mathbb{P}[\mu^n(A_{\epsilon, r}^c) > 1/r] \quad (\text{union bound over } r) \\ &\leq \sum_r r \mathbb{E}[\mu^n(A_{\epsilon, r}^c)] \quad (\text{Markov's inequality}) \\ &\leq \sum_r \sum_k r \frac{1}{n} \sum_i \mathbb{P}\left[m_i^n(T, \delta_k) > \frac{r^3}{k^2}\right] \quad (\text{union bound over } k) \\ &\leq \sum_{r, k} \frac{k^2}{r^2} \frac{1}{n} \sum_i \mathbb{E}[m_i^n(T, \delta_k)] \quad (\text{Markov's inequality}) \\ &\leq \epsilon \sum_{r, k} (rk)^{-2}. \end{aligned}$$

We conclude that $\mathbb{P}[\mu^n \notin A_\epsilon] \leq B\epsilon$ for a constant B , which provides tightness of the sequence $\{\mu^n\}$. To deduce the second statement of the lemma let $p_n(\delta) = \max_i \mathbb{P}[m_i^n(T, \delta) > m(\delta)]$ for $\delta \leq 1$. Observe that $(1/n) \sum_i \mathbb{E}[m_i^n(T, \delta)] \leq m(\delta) + 2p_n(\delta)$. Taking limits in n and then $\delta \rightarrow 0$ we see that the criterion in (2.1) holds. \square

Deterministic limits of random measures In order to show that the limit of ν^n is deterministic we will use the following criterion. Suppose ν is a random measure in $\mathcal{M}(D(\mathbb{R}_+))$. Then ν is a $\mathcal{M}(D(\mathbb{R}_+))$ -valued random variable. As such, we may first sample an outcome of ν and then sample two independent processes, X and Y , from said outcome. The joint law of the pair (X, Y) is then a probability measure on $D(\mathbb{R}_+, [0, 1]^2)$, which we call the law of two samples from ν . We may also sample an outcome from an independent copy of ν itself and then sample a process Z from that outcome. If the joint law of (X, Y) equals the joint law of (X, Z) then ν is deterministic. This is the content of the following lemma.

Lemma 3. *Let K be a compact metric space and η be a random measure on K . Let x, x' be two samples from η and let x'' be a sample from an independent copy of η . If the joint law of (x, x') equals the joint law of (x, x'') as K^2 -valued random variables, then there is a $\mu \in \mathcal{M}(K)$ such that $\eta \equiv \mu$ almost surely.*

Proof. Suppose η is defined on a probability space (Ω, Σ, P) and let $\eta(\omega)$ denote an outcome from η . As $\eta(\omega)$ is a Borel probability measure on K we may consider the integral $I_f(\omega) = \int f d\eta(\omega)$ of a continuous function $f : K \rightarrow \mathbb{R}$. Then $\omega \rightarrow I_f(\omega)$ defines a \mathbb{R} -valued random variable. The assumption that the law of two samples from η is the same as the law of samples from two independent copies of η implies that $\mathbb{E}[I_f^2] = \mathbb{E}[I_f]^2$ for every continuous f . Therefore, $\mathbb{E}[(I_f - \mathbb{E}[I_f])^2] = 0$. This implies that there is an $\Omega_f \subset \Omega$ with $P(\Omega_f) = 1$ such that $I_f(\omega) = \mathbb{E}[I_f]$ for every $\omega \in \Omega_f$.

Define a deterministic measure $\mathbb{E}[\eta] \in \mathcal{M}(K)$ by the criterion that for every continuous $f : K \rightarrow \mathbb{R}$ we have $\int f d\mathbb{E}[\eta] = \mathbb{E}[I_f]$. This defines the measure by the Reisz representation theorem as K is compact. Consider a countable dense set of continuous function $\{f_n\}$ from $K \rightarrow \mathbb{R}$ with respect to the topology of uniform convergence. Set $\Omega_\infty = \cap_n \Omega_{f_n}$. Then $P(\Omega_\infty) = 1$, and for every $\omega \in \Omega_\infty$ we have that $I_{f_n}(\omega) = \int f_n d\mathbb{E}[\eta]$ for every n .

We may approximate an arbitrary continuous function by the f_n 's. Thus, we deduce that $I_f(\omega) = \int f d\mathbb{E}[\eta]$ for every continuous $f : K \rightarrow \mathbb{R}$ so long as $\omega \in \Omega_\infty$. From the Reisz representation theorem we conclude that $\eta(\omega) = \mathbb{E}[\eta]$ for every $\omega \in \Omega_\infty$, as required. \square

3 Proof of Theorem 1

The trajectory of each particle in \mathbf{Int}^n is a continuous time simple random walk on L_n . In order to show convergence of the empirical measure ν^n of particle trajectories we will first show that it has subsequential limits by the criterion on Lemma 2. Any subsequential limit will be a random permuton process with sample paths of class Hölder(1, 1/8) due to the estimate provided in Lemma 4 below. Since the set of such functions is compact in $D(\mathbb{R}_+)$ we will then verify the criterion of Lemma 3 to conclude that any subsequential limit must be a deterministic permuton process. Lemma 5 provides the quantitate estimate used to verify this criterion. Finally, we will identify the limit as stationary Brownian motion on $[0, 1]$ via Donsker's theorem. The most technical part of the argument is the proof of Lemma 5, which is given in Section 3.1.

Lemma 4 (Tightness). *For $T < \infty$, $\delta > 0$ and any particle trajectory T_i^n of \mathbf{Int}^n ,*

$$\mathbb{P} \left[\sup_{s, t \in [0, T]; |t-s| \leq \delta} |T_i^n(t) - T_i^n(s)| > \delta^{1/8} \right] \leq 10^3 T \left(\delta^{1/2} + \frac{\delta^{-1/2}}{n^2} \right).$$

Consequently, the empirical measures ν^n of the particle trajectories have subsequential limits and all limit points are random permuton processes with sample paths of class Hölder(1, 1/8) almost surely.

Lemma 5 (Concentration). *For any two particles $i \neq j$ there exists coupled random trajectories $T_1(t)$, $T_2(t)$ and $T_3(t)$ for $t \geq 0$ with the following properties. The pair (T_1, T_2) has the joint law of (T_i^n, T_j^n) , the trajectories of particles i and j in \mathbf{Int}^n . T_3 is independent of T_1 and has the law of T_j^n . Finally, for some universal constant C ,*

$$\mathbb{P} \left[\sup_{0 \leq t \leq T} |T_2(t) - T_3(t)| > n^{-\frac{1}{4}} \right] \leq \frac{C T^{1/2}}{n^{1/2}}.$$

A covering argument In order to prove Lemmas 4 and 5 we interpret the law of trajectories of particles in \mathbf{Int}^n as projections of particle trajectories of the interchange process on \mathbb{Z} . Consider the map $\pi_n : \mathbb{Z} \rightarrow V(L_n)$ defined as follows. First, map an integer x to $x \pmod{2n}$ with coset representative chosen from the set $\{1, \dots, 2n\}$. Second, map the coset representative y to $y \in V(L_n)$ if $1 \leq y \leq n$, and otherwise, map y to $2n + 1 - y \in V(L_n)$. The map π_n is the composition of the first map followed by the second. Note that π_n preserves vertex adjacencies: $\{\pi_n(x), \pi_n(x+1)\}$ is an edge of L_n for every $x \in \mathbb{Z}$. In particular, π_n is Lipschitz:

$$|\pi_n(a) - \pi_n(b)| \leq |a - b| \text{ for every } a, b \in \mathbb{Z}.$$

Consider the continuous time interchange process on \mathbb{Z} where particles move according to edge firings of independent Poisson processes $\{\text{Poi}_{\{x, x+1\}}\}$ for $x \in \mathbb{Z}$, each firing at rate 1/2. Let $S_i(t)$ be the trajectory of particle i . Then $(S_i(t); t \geq 0)$ has the law of continuous time simple random walk (SRW) on \mathbb{Z} started from i . The continuous time SRW may be interpreted at the law of $\{i + Z_{N(t)}; t \geq 0\}$, where $Z_0 = 0, Z_1, \dots$ is a

discrete time SRW on \mathbb{Z} and $(N(t); t \geq 0)$ is a rate 1 Poisson process on \mathbb{R} that is independent of Z_0, Z_1, \dots . It is easy to see from this that

$$\mathbb{E} [|S_i(t_2) - S_i(t_1)|^4] = 3(t_2 - t_1)^2 + t_2 - t_1 \text{ for all times } 0 \leq t_1 \leq t_2 < \infty.$$

Observe that for any finite set of particles $1 \leq i_1 < \dots < i_k \leq n$, the joint law of the trajectories $(\mathbf{Int}_t^n(i_1), \dots, \mathbf{Int}_t^n(i_k); t \geq 0)$ is the same as the joint law of $(\pi_n(S_{i_1}(t)), \dots, \pi_n(S_{i_k}(t)); t \geq 0)$.

Let $(\mathcal{F}(t); t \geq 0)$ denote the natural filtration for the above interchange process on \mathbb{Z} . Namely, $\mathcal{F}(t)$ is generated by the sigma-algebras $\mathcal{F}_{\{x, x+1\}}(t) = \sigma(\text{Poi}_{\{x, x+1\}}[0, t])$ over all $x \in \mathbb{Z}$. Then $(\mathcal{F}(t); t \geq 0)$ is a right continuous filtration and each particle trajectory S_i is a SRW on \mathbb{Z} adapted to the filtration.

Proof of Lemma 4 (Tightness)

Proof. The cover map π_n allows us to bound the modulus of continuity of $\mathbf{Int}_t^n(i)$ by way of the modulus of continuity of S_i . For each i , the process $S_i(t) - S_i(0)$ is a martingale w.r.t. the filtration $(\mathcal{F}(t); t \geq 0)$. By Doob's maximal inequality for martingales (continuous time with càdlàg paths) we have that for all particles i and times $0 \leq t_1 \leq t_2 < \infty$,

$$\mathbb{P} \left[\sup_{t_1 \leq s \leq t_2} |S_i(s) - S_i(t_1)| > \lambda \right] \leq \frac{16 \mathbb{E} [|S_i(t_2) - S_i(t_1)|^4]}{\lambda^4} \leq \frac{48 |t_2 - t_1|^2 + 16 |t_2 - t_1|}{\lambda^4}. \quad (3.1)$$

The law of the trajectory T_i^n of \mathbf{Int}_t^n is the same as the law of $(\pi_n(S_i(n^2 t))/n; t \geq 0)$. As π_n is Lipschitz we have from (3.1) that for all particles i , times $t \geq 0$, and $0 \leq \delta \leq 1$,

$$\begin{aligned} \mathbb{P} \left[\sup_{t \leq s \leq t+\delta} |T_i^n(s) - T_i^n(t)| > \delta^{1/8} \right] &= \mathbb{P} \left[\sup_{t \leq s \leq (t+\delta)} \left| \frac{\pi_n(S_i(n^2 s))}{n} - \frac{\pi_n(S_i(n^2 t))}{n} \right| > \delta^{1/8} \right] \\ &\leq \mathbb{P} \left[\sup_{t \leq s \leq (t+\delta)} |S_i(n^2 s) - S_i(n^2 t)| > n \delta^{1/8} \right] \\ &\leq 2^6 \left(\delta^{3/2} + \frac{\delta^{1/2}}{n^2} \right). \end{aligned} \quad (3.2)$$

Fix $T < \infty$ and set $m_i(\delta) = \sup_{s, t \in [0, T]; |s-t| \leq \delta} |T_i^n(t) - T_i^n(s)|$. Note that m_i dominates the càdlàg modulus of continuity of T_i^n by considering the partition of $[0, T]$ into equally spaced points of mesh size δ . We have that

$$m_i(\delta) \leq 2 \max_{1 \leq j \leq \lceil T/\delta \rceil} \sup_{(j-1)\delta \leq s, t \leq j\delta} |T_i^n(t) - T_i^n(s)|.$$

Employing an union bound over j and utilizing (3.2) we deduce that

$$\max_{1 \leq i \leq n} \mathbb{P} [m_i(\delta) > 2\delta^{1/8}] \leq 2^6 \left\lceil \frac{T}{\delta} \right\rceil (\delta^{3/2} + \delta^{1/2} n^{-2}) \leq 2^7 T (\delta^{1/2} + \frac{\delta^{-1/2}}{n^2}).$$

From Lemma 2 we conclude that the random measures $\{\nu^n\}$ with times restricted to $[0, T]$ has subsequential weak limits converging to random measures on $D([0, T])$. As this holds for every T it follows that $\{\nu^n\}$ has weak subsequential limits converging to random measure on $D(\mathbb{R}_+)$. Any limit point is a random permuton process $(\mathbf{X}(t); 0 \leq t \leq T)$. The estimate (3.2) readily implies that the outcomes of \mathbf{X} are permuton processes with sample paths of class Hölder(1, 1/8). This completes the proof of Lemma 4. \square

Proof of Theorem 1 Lemma 5 is proved in Section 3.1 below. We will finish the proof of Theorem 1 from the two key lemmas. Lemma 4 shows that ν^n has subsequential limits that are supported on permuton processes with Hölder(1,1/8) sample paths. Let \mathbf{X} be such a limit point, which we will conclude is a deterministic permuton process. By way of Lemma 3 it suffices to show that the joint law of two samples from \mathbf{X} is the same as the law of samples from two independent copies of \mathbf{X} . After reducing to a subsequence we may assume that ν^n converges to \mathbf{X} .

Employing Lemma 5, for every $1 \leq i \leq n$ and $j \neq i$ we find coupled trajectories $(T_i^n, T_j^n, T_{i,j}^n)$ such that (T_i^n, T_j^n) has the joint law of the trajectories of particles i and j in \mathbf{Int}^n while $T_{i,j}^n$ has the law of T_j^n and is independent of T_i^n . We also have that for some universal constant C ,

$$\mathbb{P} \left[\left\| T_j^n(t) - T_{i,j}^n(t) \right\|_{L^\infty[0,T]} > \frac{2}{n^{1/4}} \right] \leq \frac{C\sqrt{T}}{\sqrt{n}}. \quad (3.3)$$

For every $1 \leq i, j \leq n$ set $X_i = T_i^n$, $Y_j = T_j^n$ if $j \neq i$ and $Y_i = X_i$, and $Z_{i,j} = T_{i,j}^n$ if $i \neq j$ and $Z_{i,i} = X_i$. Let $I, J \sim \text{Uniform}[\{1, \dots, n\}]$ be independent random indices that are also independent of \mathbf{Int}^n . We omit writing the dependence of I and J on n . The pair (X_I, Y_J) converges to the law of two samples from \mathbf{X} and the pair $(X_I, Z_{I,J})$ converges to the law of samples from two independent copies of \mathbf{X} . It suffices to show that $\|(X_I, Y_J) - (X_I, Z_{I,J})\|_{L^\infty[0,T]}$ converges to zero in probability to conclude that \mathbf{X} is deterministic. To that end, note that

$$\begin{aligned} \mathbb{P} \left[\|Y_J - Z_{I,J}\|_{L^\infty[0,T]} > 2n^{-1/4} \right] &\leq \mathbb{P} \left[\|Y_J - Z_{I,J}\|_{L^\infty[0,T]} > 2n^{-1/4}; I \neq J \right] + \frac{1}{n} \\ &= \frac{1}{n^2 - n} \sum_{i \neq j} \mathbb{P} \left[\|Y_j - Z_{i,j}\|_{L^\infty[0,T]} > 2n^{-1/4} \right] + \frac{1}{n} \\ &\leq \max_{i \neq j} \mathbb{P} \left[\|Y_j - Z_{i,j}\|_{L^\infty[0,T]} > 2n^{-1/4} \right] + \frac{1}{n} \\ &\leq \frac{C\sqrt{T}}{\sqrt{n}} + \frac{1}{n}. \end{aligned}$$

This concludes the proof that all limit points of ν^n are deterministic. Also, the limit points are the limit points of X_I , that is, the weak limit points of the trajectory of a random particle considered as a random function in $D(\mathbb{R}_+)$. The law of X_I is the same as that of the process

$$\frac{2}{n} \pi_n(I + S(n^2 t)) - 1 \quad \text{for } t \geq 0,$$

where $t \rightarrow S(t)$ is SRW on \mathbb{Z} started from the origin and independent of I .

Consider the map $\pi : \mathbb{R} \rightarrow [0, 1]$ defined as follows. First, map any real number x to its unique coset representative y in $\mathbb{R}/2\mathbb{Z}$ with $y \in (0, 2]$. Then map y to itself if $0 < y \leq 1$, and otherwise, map y to $2 - y \in [0, 1)$. The map π is Lipschitz and $|\frac{\pi_n(k)}{n} - \pi(\frac{k}{n})| \leq \frac{1}{n}$ for every $k \in \mathbb{Z}$. This estimate between π_n and π implies that

$$\left\| \left(\frac{\pi_n(I + S(n^2 t))}{n} \right) - \left(\pi \left(\frac{I + S(n^2 t)}{n} \right) \right) \right\|_{L^\infty[0,T]} \leq \frac{1}{n}.$$

The estimate above implies that any limit point of X_I must be stationary Brownian motion on $[0, 1]$ so long as the process $(\pi(\frac{I + S(n^2 t)}{n}); t \geq 0)$ converges to stationary Brownian motion on $[0, 1]$. Donsker's Theorem implies that the process $(\frac{S(n^2 t)}{n}; t \geq 0)$ converges weakly on $D(\mathbb{R}_+, \mathbb{R})$ to standard Brownian

motion $(B(t); t \geq 0)$. As I is independent of S we deduce that the process

$$\left(\frac{I + S(n^2 t)}{n}; t \geq 0 \right) \rightarrow (U + B(t); t \geq 0) \text{ weakly,}$$

where $U \sim \text{Uniform}[0, 1]$ is independent of B . Since π is Lipschitz and thus continuous, we have that $(\pi(\frac{I + S(n^2 t)}{n}); t \geq 0)$ converges to $(\pi(U + B(t)); t \geq 0)$. The latter process is stationary Brownian motion on $[0, 1]$, which is Brownian motion started from an uniform random point in $[0, 1]$ and reflected off of the lines $y = 0$ and $y = 1$.

We thus conclude that $\{\nu^n\}$ is tight and any limit point \mathbf{X} must be stationary Brownian motion on $[0, 1]$. This implies that $\{\nu^n\}$ converges to stationary Brownian motion on $[0, 1]$.

3.1 Concentration of the interchange process: Proof of Lemma 5

We will prove Lemma 5 by utilizing the covering of \mathbf{Int}^n by the interchange process on \mathbb{Z} introduced earlier. We will first construction appropriate coupled trajectories for the interchange process on \mathbb{Z} and then project them to \mathbf{Int}^n using the covering map π_n .

Fix particles $1 \leq i < j \leq n$ and let S_1 and S_2 denote the trajectories S_i and S_j of particles i and j for the interchange process on \mathbb{Z} , respectively. We define a new trajectory S_3 of particle j that will be independent of S_1 but coupled with S_2 so that $\|S_2 - S_3\|_{L^\infty[0, T]}$ is of order \sqrt{T} . Roughly speaking, S_3 will take the same steps as S_2 except between times when $|S_1 - S_2| = 1$, where S_3 will move independently as a SRW on \mathbb{Z} . More precisely, consider the stopping times $0 = \tau_0^+ < \tau_1 < \tau_1^+ < \dots$, where

$$\tau_j = \inf\{t \geq \tau_{j-1}^+ : |S_1(t) - S_2(t)| = 1\} \quad \text{and} \quad \tau_j^+ = \inf\{t \geq \tau_j : |S_1(t) - S_2(t)| > 1\}.$$

As SRW on \mathbb{Z} is recurrent the τ_j s and τ_j^+ s are all finite and satisfy $\tau_1 < \tau_1^+ < \tau_2 < \dots$ with probability 1. They are also strict stopping times in the sense that the events $\{\tau_j \leq t\}$ and $\{\tau_j^+ \leq t\}$ are $\mathcal{F}(t)$ -measurable. Set $\mathcal{C} = \cup_j [\tau_j, \tau_j^+)$. Then $|S_1(t) - S_2(t)| = 1$ precisely for times $t \in \mathcal{C}$.

In order to define the process S_3 we introduce a new collection of independent Poisson processes $\{\text{Poi}'_{\{x, x+1\}}\}$ for $x \in \mathbb{Z}$ that are also independent of the processes $\{\text{Poi}_{\{x, x+1\}}\}$ used to define the interchange process on \mathbb{Z} . We may assume that the edge firings of the new Poisson processes and the old one are all distinct. Let $S_3(0) = j$, and for times $t \in \mathcal{C}$ let the trajectory S_3 move according to the edge firings of $\{\text{Poi}'_{\{x, x+1\}}\}$. More precisely, if $t \in \mathcal{C}$ and the edge $\{x, x+1\}$ fires at time t in $\text{Poi}'_{\{x, x+1\}}$ with $S_3(t-) = x$ then let $S_3(t) = x+1$. For other times t the particle at $S_3(t)$ moves according to the direction of the particles at $S_2(t)$. Namely, if $S_2(t)$ changes to $S_2(t) + 1$ or $S_2(t) - 1$ after encountering an edge firing at time t then $S_3(t)$ also moves to $S_3(t) + 1$ or $S_3(t) - 1$, respectively, in accordance to $S_2(t)$'s direction. In particular, $|S_3(t) - S_2(t)|$ changes only for times $t \in \mathcal{C}$. The process S_3 is independent of S_1 and the law of S_3 is that of a SRW on \mathbb{Z} started from j .

We now bound $\mathbb{P}[\|S_3 - S_2\|_{L^\infty[0, T]} > \lambda]$. The processes $S_2(t)$ and $S_3(t)$ are both martingales with respect to the natural filtration induced by $\{\text{Poi}_{\{x, x+1\}}, \text{Poi}'_{\{x, x+1\}}\}_{x \in \mathbb{Z}}$. Therefore, $S_3(t) - S_2(t)$ is a martingale with $S_3(0) - S_2(0) = 0$. Doob's maximal inequality gives

$$\mathbb{P}\left[\sup_{0 \leq t \leq T} |S_3(t) - S_2(t)| > \lambda\right] \leq \frac{4}{\lambda^2} \mathbb{E}[|S_3(T) - S_2(T)|^2]. \quad (3.4)$$

We will show that $\mathbb{E}[|S_3(T) - S_2(T)|^2]$ is of order $O(\sqrt{T})$ in the following.

Consider the random variable

$$J = \inf\{j \geq 1 : \tau_j^+ > T\}. \quad (3.5)$$

The number of times particles i and j are distance 1 apart within the time interval $[0, T]$ is either J or $J - 1$. The random variable J is $\mathcal{F}(T)$ -measurable because the event $\{J \geq j\}$ equals $\mathcal{F}(T)$ -measurable event that particles i and j have been at distance 1 from each other at least j times within the time period $[0, T]$. Moreover, the event $\{J \geq j\}$ is $\mathcal{F}(\tau_j)$ -measurable because $\{J \geq j\} \cap \{\tau_j \leq t\}$ is the event that particles i and j have been distance 1 apart at least j times in the time interval $[0, \min\{t, T\}]$.

Lemma 6. *For $T \geq 1$, the expectation $\mathbb{E}[J] \leq 10\sqrt{T}$.*

Proof. Consider the process Z defined by $Z(t) = |S_2(t) - S_1(t)| - 1$, which takes values in $\mathbb{Z}_{\geq 0} = \{0, 1, 2, \dots\}$. The law of Z is of a random walk on $\mathbb{Z}_{\geq 0}$ with edges firing as follows. The edges $\{x, x+1\}$ for $x \geq 0$ fire at rate 1 and there is a self loop at $\{0\}$ that fires at rate $1/2$. Let V be the number of times Z visits the origin $\{0\}$ in the time interval $[0, T]$. Observe that $J \leq V + 1$ because the times when $Z(t) = 0$ are the times when particles i and j are distance 1 apart in the interchange process on \mathbb{Z} .

Consider the SRW on $\mathbb{Z}_{\geq 0}$ with all edges $\{x, x+1\}$ firing at rate 1 and also a self loop at $\{0\}$ firing at rate $1/2$. If we start the walk at $j - i - 1$ then V is stochastically dominated by the number of visits to the origin of this SRW in the time period $[0, T]$. Indeed, the edge firings of the self loop for the process Z is stochastically dominated by the edge firings of this SRW in the sense that the two processes can be coupled such that the edge firings of the former can be made to be contained in the edge firings of the latter. This follows because if every point of a Poisson process with rate 1 is deleted independently with probability $1/2$ then it results in a Poisson process of rate $1/2$ (known as the thinning property of Poisson processes). Hence, it suffices to bound the expected number of visits to the origin in time period $[0, T]$ of a SRW on $\mathbb{Z}_{\geq 0}$ with all edges firing as rate 1.

We may cover the above graph on $\mathbb{Z}_{\geq 0}$, with the self loop, via a map $\hat{\pi} : \mathbb{Z} \rightarrow \mathbb{Z}_{\geq 0}$ that sends the law of SRW on \mathbb{Z} to the law of SRW on $\mathbb{Z}_{\geq 0}$. This is entirely analogous to the covering maps $\pi_n : \mathbb{Z} \rightarrow L_n$ introduced earlier. Visits to $\{0\}$ of the SRW on $\mathbb{Z}_{\geq 0}$ then corresponds to visits to the set $\{0, 1\}$ for the SRW on \mathbb{Z} . The expected number of visits of SRW on \mathbb{Z} , started from any vertex, to the set $\{0, 1\}$ in time period $[0, T]$ is at most $6\sqrt{T}$ by Lemma 9. This implies that $\mathbb{E}[J] \leq 10\sqrt{T}$. \square

We now proceed to bound $\mathbb{E}[|S_3(T) - S_2(T)|^2]$. As $S_3(t) - S_2(t)$ changes only for times $t \in \mathcal{C}$, we have

$$\begin{aligned} S_3(T) - S_2(T) &= \sum_j [(S_3(\tau_j^+) - S_3(\tau_j)) - (S_2(\tau_j^+) - S_2(\tau_j))] \cdot \mathbf{1}_{\{j < J\}} + \\ &\quad + [(S_3(T) - S_3(\tau_J)) - (S_2(T) - S_2(\tau_J))] \cdot \mathbf{1}_{\{T > \tau_J\}}. \end{aligned} \quad (3.6)$$

We consider the distributions of the random variables in the summands above in order to calculate their mean and variance.

We claim that $[S_2(\tau_j^+) - S_2(\tau_j)] \cdot \mathbf{1}_{\{j < J\}}$ is $\mathcal{F}(T)$ -measurable for every j . Indeed, the event $\{j < J\}$ implies $\{\tau_j^+ \leq T\}$, which then implies that

$$[S_2(\tau_j^+) - S_2(\tau_j)] \mathbf{1}_{\{j < J\}} = [S_2(\tau_j^+) - S_2(\tau_j)] \mathbf{1}_{\{\tau_j^+ \leq T\}} \mathbf{1}_{\{j < J\}}.$$

The r.h.s. above is $\mathcal{F}(T)$ -measurable as both $[S_2(\tau_j^+) - S_2(\tau_j)] \mathbf{1}_{\{\tau_j^+ \leq T\}}$ and $\mathbf{1}_{\{j < J\}}$ are $\mathcal{F}(T)$ -measurable. Similarly, $(S_2(T) - S_2(\tau_J)) \mathbf{1}_{\{T > \tau_J\}}$ is $\mathcal{F}(T)$ -measurable. Indeed, observe that $\{T > \tau_J\}$ is an $\mathcal{F}(T)$ -measurable event since it means that particles i and j are distance 1 apart at time T .

Now consider the distribution of $[S_3(\tau_j^+) - S_3(\tau_j)] \mathbf{1}_{\{j < J\}}$ conditional on $\mathcal{F}(T)$. By design, this conditional

distribution is the displacement of a SRW on \mathbb{Z} from time 0 to time $\tau_j^+ - \tau_j$. Hence,

$$\begin{aligned}\mathbb{E} [S_3(\tau_j^+) - S_3(\tau_j)] \mathbf{1}_{\{j < J\}} \mid \mathcal{F}(T) &= 0 \quad \text{and} \\ \mathbb{E} [S_3(\tau_j^+) - S_3(\tau_j)]^2 \mathbf{1}_{\{j < J\}} \mid \mathcal{F}(T) &= (\tau_j^+ - \tau_j) \mathbf{1}_{\{j < J\}}.\end{aligned}$$

Similarly, $[S_3(T) - S_3(\tau_J)] \mathbf{1}_{\{T > \tau_J\}}$ has mean 0 and variance $(T - \tau_J) \mathbf{1}_{\{T > \tau_J\}}$ conditional on $\mathcal{F}(T)$.

In order to calculate $\mathbb{E} [|S_3(T) - S_2(T)|^2]$ we first condition over $\mathcal{F}(T)$, use the representation (3.6), and use the observations above about the conditional expectations given $\mathcal{F}(T)$ to conclude that

$$\begin{aligned}\mathbb{E} [|S_3(T) - S_2(T)|^2] &= \mathbb{E} \left[\sum_{j < J} (\tau_j^+ - \tau_j) + (T - \tau_J) \mathbf{1}_{\{T > \tau_J\}} \right] \\ &\quad + \mathbb{E} \left[\left(\sum_{j < J} S_2(\tau_j^+) - S_2(\tau_j) + (S_2(T) - S_2(\tau_J)) \mathbf{1}_{\{T > \tau_J\}} \right)^2 \right].\end{aligned}\tag{3.7}$$

In the following two lemmas we bound the first and second expectations on the r.h.s. of (3.7).

Lemma 7. $\mathbb{E} \left[\sum_{j < J} (\tau_j^+ - \tau_j) + (T - \tau_J) \mathbf{1}_{\{T > \tau_J\}} \right] \leq 10\sqrt{T}$.

Proof. Define

$$M_n = \sum_{j=1}^n (\tau_j^+ - \tau_j - 1).$$

We make use of the strong Markov property of the process $\vec{S} = ((S_1(t), S_2(t)); t \geq 0)$. By the strong Markov property the $\tau_j^+ - \tau_j$ are independent and identically distributed because $\tau_j^+ - \tau_j$ is a stopping time for $(\vec{S}(t + \tau_j) - \vec{S}(\tau_j))$. Moreover, $\tau_1^+ - \tau_1$ is distributed as an exponential random variable of rate 1. Indeed, its distribution is the first time particles 0 and 1 from the interchange process on \mathbb{Z} move farther than distance 1 apart, which is the minimum of two independent exponential random variables of rate 1/2. These observations imply that $\{M_n\}$ is a mean zero martingale with respect to the filtration $(\mathcal{F}(\tau_n^+); n \geq 0)$. The random variable J is a stopping time for this filtration since the event $\{J \geq j\} \in \mathcal{F}(\tau_j) \subset \mathcal{F}(\tau_j^+)$. Assume that $T \geq 1$. As $\mathbb{E}[J] \leq 10\sqrt{T}$ by Lemma 6, the Optional Stopping Theorem implies $\mathbb{E}[M_J] = 0$. As $(T - \tau_J) \mathbf{1}_{\{T > \tau_J\}} \leq \tau_J^+ - \tau_J$, we have

$$\sum_{j < J} (\tau_j^+ - \tau_j) + (T - \tau_J) \mathbf{1}_{\{T > \tau_J\}} \leq M_J + J.$$

Therefore,

$$\mathbb{E} \left[\sum_{j < J} (\tau_j^+ - \tau_j) + (T - \tau_J) \mathbf{1}_{\{T > \tau_J\}} \right] \leq \mathbb{E}[J] \leq 10\sqrt{T}.$$

□

Lemma 8.

$$\mathbb{E} \left[\left(\sum_{j < J} S_2(\tau_j^+) - S_2(\tau_j) + (S_2(T) - S_2(\tau_J)) \mathbf{1}_{\{T > \tau_J\}} \right)^2 \right] \leq 100\sqrt{T}.$$

Proof. Note that $|S_2(T) - S_2(\tau_J)|\mathbf{1}_{\{T > \tau_J\}} \leq 1$. From the inequality $(a + b)^2 \leq 2(a^2 + b^2)$ applied with $b = S_2(T) - S_2(\tau_J)\mathbf{1}_{\{T > \tau_J\}}$ we deduce that

$$\left(\sum_{j < J} S_2(\tau_j^+) - S_2(\tau_j) + (S_2(T) - S_2(\tau_J))\mathbf{1}_{\{T > \tau_J\}} \right)^2 \leq 2 \left(\sum_{j < J} S_2(\tau_j^+) - S_2(\tau_j) \right)^2 + 2. \quad (3.8)$$

We claim that $\left\{ \sum_{j \leq n} (S_2(\tau_j^+) - S_2(\tau_j)) \right\}_{n \geq 1}$ is a mean zero Martingale adapted to the filtration $\{\mathcal{F}(\tau_n^+)\}$ with the increment $S_2(\tau_j^+) - S_2(\tau_j)$ being independent of $\mathcal{F}(\tau_{j-1}^+)$. To see this note that the strong Markov property of the process $\vec{S} = (S_1, S_2)$ implies that $S_2(\tau_j^+) - S_2(\tau_j)$ is independent of $\mathcal{F}(\tau_{j-1}^+)$. The Optional Stopping Theorem gives $\mathbb{E}[S_2(\tau_j^+) - S_2(\tau_j)] = 0$ because $\tau_j^+ - \tau_j$ is a stopping time for the process $(\vec{S}(t + \tau_j) - \vec{S}(\tau_j); t \geq 0)$ and it has finite expectation. Finally, observe that $|S_2(\tau_j^+) - S_2(\tau_j)| \leq 2$. Indeed, suppose two particles a and b are distance 1 apart on the interchange process during the time interval $[t_1, t_2]$. If they drift farther apart at time t_2 then the absolute displacement of either a or b from its respective position at time t_1 to its position at time t_2 is at most 2. These three observations imply the claim. In particular,

$$M'_n = \left(\sum_{j \leq n} S_2(\tau_j^+) - S_2(\tau_j) \right)^2 - \sum_{j \leq n} \mathbb{E}[(S_2(\tau_j^+) - S_2(\tau_j))^2]$$

is also a Martingale. The Optional Stopping Theorem now implies $\mathbb{E}[M'_{J-1}] = 0$. Combining this with the estimate (3.8) we get that

$$\begin{aligned} \mathbb{E} \left[\left(\sum_{j < J} S_2(\tau_j^+) - S_2(\tau_j) + (S_2(T) - S_2(\tau_J))\mathbf{1}_{\{T > \tau_J\}} \right)^2 \right] &\leq \\ &2\mathbb{E} \left[\left(\sum_{j < J} S_2(\tau_j^+) - S_2(\tau_j) \right)^2 \right] + 2 = \\ &2\mathbb{E} \left[\sum_{j < J} (S_2(\tau_j^+) - S_2(\tau_j))^2 \right] + 2 \leq \\ &8\mathbb{E}[J - 1] + 2 \leq 100\sqrt{T}. \end{aligned}$$

□

From (3.7), Lemma 7 and Lemma 8, we conclude that $\mathbb{E}[|S_3(T) - S_2(T)|^2] \leq C\sqrt{T}$ for some universal constant C so long as $T \geq 1$. Then (3.4) implies that

$$\mathbb{P} \left[\sup_{0 \leq t \leq T} |S_3(t) - S_2(t)| > \lambda \right] \leq \frac{C \max\{T^{1/2}, 1\}}{\lambda^2}.$$

After applying the cover map π_n , we get that the processes $\pi_n(S_1), \pi_n(S_2)$ and $\pi_n(S_3)$ have the property that the first two have the law of the non-rescaled trajectories of particles i and j in \mathbf{Int}^n while the third has the law of the non-rescaled trajectory of particle j and is independent of the first. As π_n is Lipschitz we get that

$$\mathbb{P} \left[\sup_{0 \leq t \leq n^2 T} |\pi_n(S_2(t)) - \pi_n(S_3(t))| > \lambda \right] \leq \frac{C \max\{T^{1/2}, 1\} n}{\lambda^2}.$$

Setting $T_k(t) = \pi_n(S_k(t))/n$ for $k = 1, 2, 3$ and $\lambda = n^{3/4}$ we get the conclusion of Lemma 5.

Lemma 9 (SRW visits to origin). *Let $S = (S(t); t \geq 0)$ be continuous time SRW on \mathbb{Z} where all edges fire at rate 1 and $S(0) = x$ for some fixed $x \in \mathbb{Z}$. Let $V(T)$ be the number of visits of S to the origin in the time period $[0, T]$. Then $\mathbb{E}[V(T)] \leq 3\sqrt{T}$. In particular, the expected number of visits to a set of vertices $\{y_1, \dots, y_j\}$ within the time interval $[0, T]$ is at most $3j\sqrt{T}$.*

Proof. Let $S_0 = x, S_1, S_2, \dots$ be a discrete time SRW on \mathbb{Z} . Let P be a Poisson process of rate 2 that is independent of S_0, S_1, \dots . Let $N(t)$ be the number of points of P in $[0, t]$. The law of the continuous time SRW on \mathbb{Z} with edges firing at rate 1 is $(S_{N(t)}; t \geq 0)$. Let V_n be the number of visits of S_0, S_1, \dots to the origin from time 0 to time n . Then $V(T)$ equals $V_{N(T)}$ in distribution. If we show that $\mathbb{E}[V_n] \leq 2\sqrt{n}$, then $N(T)$ being independent of V_0, V_1, \dots implies that

$$\mathbb{E}[V(T)] = \mathbb{E}[\mathbb{E}[V_{N(T)} | N(T)]] \leq 2\mathbb{E}[\sqrt{N(T)}] \leq 2\mathbb{E}[N(T)]^{1/2} = 2\sqrt{2}\sqrt{T}, \text{ as required.}$$

Note that $\mathbb{E}[V_n] = \sum_{k=0}^n \mathbb{P}[S_k = 0]$. We may assume that $x \geq 0$ due to symmetry. $\mathbb{P}[S_k = 0]$ is non-zero only if k and x have the same parity and $k \geq x$. Then,

$$\mathbb{P}[S_k = 0] = \binom{k}{(k+x)/2} 2^{-k} \leq \binom{k}{\lfloor \frac{k}{2} \rfloor} 2^{-k} \leq \frac{1}{\sqrt{k}}.$$

The last estimate follows from Stirling's approximation. Therefore,

$$\mathbb{E}[V_n] \leq 1 + \sum_{k=1}^n \frac{1}{\sqrt{k}} \leq 2\sqrt{n}.$$

The last statement of the lemma follows from observing that the number of visits to a vertex y of a SRW started from x has the same law as the number of visits to the origin of a SRW started from $x - y$. \square

4 The hydrodynamic limit

The simple symmetric exclusion process (SSEP) is a particle system on L_n evolving as follows [2, chapter 2.2]. There are two types of particles - black particles, or occupied sites, and white particles, also called free sites. The initial configuration of black particles is some subset $\eta_0 \in \{0, 1\}^{V(L_n)}$, where $\eta_0(i) = 1$ indicates a black particle at vertex i . The particles then swap positions according to the dynamics of the interchange process. If η_t is the configuration of black particles at time t then SSEP is the $\{0, 1\}^{V(L_n)}$ -valued process $(\eta_t; t \geq 0)$. Let $|\eta_0| = \sum_i \eta_0(i)$ be the number of black particles. The empirical measure of SSEP at time t is the following random probability measure on $[0, 1]$:

$$\rho_n(t) = \frac{1}{|\eta_0|} \sum_{i=1}^n \delta_{\{\frac{i}{n}\}} \cdot \eta_{n^2 t}(i). \quad (4.1)$$

Suppose the SSEP evolves according to the particle dynamics of the interchange process \mathbf{Int}^n . Let T_j^n be the rescaled trajectory of particle j in \mathbf{Int}^n . The vertices occupied by the black particles at time $n^2 t$ are given by the values of $\frac{n}{2} \cdot T_j^n(t)$ for those j such that $\eta_0(j) = 1$. Therefore,

$$\rho_n(t) = \frac{1}{|\eta_0|} \sum_{i=1}^n \delta_{T_j^n(t)} \cdot \eta_0(j).$$

Let $J_n \sim \text{Uniform}[\{j : \eta_0(j) = 1\}]$ denote a random choice of black particle. Conditional on \mathbf{Int}^n the measure $\rho_n(t)$ is the law of $T_{J_n}^n(t)$ over the random J_n . The measure valued process $(\rho_n(t); t \geq 0)$ is the law of the trajectory of a random black particle in \mathbf{Int}^n . The argument of the previous sections implies that $(\rho_n(t); t \geq 0)$ has the same weak limit as the limiting law of the process $(\pi(\frac{J_n + S(n^2t)}{n}); t \geq 0)$ considered as a random function in $D(\mathbb{R}_+)$. (Recall that $(S(t); t \geq 0)$ is SRW on \mathbb{Z} that is independent of all the J_n s, and π is the covering map from \mathbb{R} onto $[0, 1]$.)

Suppose that the law of J_n/n , which is $\rho(0)$ rescaled onto $[0, 1]$, converges weakly to the law of a random variable $X_0 \in [0, 1]$. As each J_n is independent of the SRW S , the law of the process

$$\left(\pi \left(\frac{J_n + S(n^2t)}{n} \right); t \geq 0 \right) \rightarrow \left(\pi(X_0 + B(t)); t \geq 0 \right) \text{ weakly in } D(\mathbb{R}_+),$$

where X_0 is independent of standard Brownian motion B . The latter process has the law of standard Brownian motion started from X_0 and then reflected off of the lines $y = 0$ and $y = 1$.

Consequently, for any t , the random measure $\rho_n(t)$ converges weakly to the deterministic measure induced by the law of $\pi(X_0 + B(t))$. This is the hydrodynamic limit of SSEP, describing the limiting empirical density of the black particles [2, chapter 4].

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